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Low-energy vortex dynamics in Abelian Higgs systems

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Abstract. The low-energy dynamics of the vortices of the Abelian Chern–Simons–Higgs system is investigated from the adiabatic approach. The difficulties involved in treating the field evolution as motion on the vortex moduli space in this system are shown. Another two generalized Abelian Higgs systems are discussed with respect to their vortex dynamics at the adiabatic limit. The method works well, and we find bound states in the first model and scattering at right angles in the second system.

1 Introduction

Since their discovery by Nielsen and Olesen [1], the vortex solutions present in the Abelian Higgs (AH) model have been used in a variety of contexts beyond their original purpose as vehicles of the strong forces. They have been found useful, for example, in describing cosmic strings; also, because the energy of static configurations in the AH model can be interpreted as the free energy of the Ginzburg–Landau theory for superconducting materials, these topological solutions correspond to the magnetic flux tubes appearing in type II superconductors. The spectrum of potentially relevant vortices in condensed-matter physics has recently been broadened by the discovery of a new class of outstanding cousins of the AH model solutions: the topological and non-topological solitons arising in several Chern–Simons–Higgs (CSH) gauge systems. The need to include a Chern–Simons term in the treatment of three-dimensional gauge theories was first advocated by Jackiw and Templeton [2], who were studying the radiative corrections to spinorial electrodynamics. The most remarkable effects of this term are the generation of a topological photon mass compatible with gauge invariance [2,3], and the statistical transmutation of particles coupled to the gauge field [4]. The Higgs mechanism in Maxwell–Chern–Simons electrodynamics was first investigated in [5], but although there are vortices in this system, they are not self-dual [6]. The simplest way of achieving a self-dual limit is to renounce the Maxwell term and use the effective long-wavelength model introduced in [7,8]. Self-duality with the Maxwell term is also possible, but then supplementary scalar fields become necessary [9]. The CSH vortices could provide a theoretical model for describing physically distinguished objects such as Laughlin quasiparticles or quasiholes [10] and the vortices of the still poorly understood high-T_c superconductors [11]. For this reason, any insight into their interactions and dynamical properties is of interest.

The non-linear nature of field equations having soliton-like solutions makes it almost impossible to study the dynamics of topological defects in full detail. A brilliant idea from Manton [12] allows in some cases an analytical approach to the problem: He showed that the low-energy scattering of Bogonolny-Prasad-Sommerfeld (BPS) monopoles can be traced back to geodesic motion in the moduli space of these self-dual solutions for fixed magnetic charge. The method has been generalized by Manton himself and others according to the following scheme: The adiabatic limit in the dynamics of topological defects is given by a Lagrangian system describing the motion of a particle in the moduli space of self-dual solutions $\mathcal{M}_n$. The mechanical kinetic energy comes from the terms which are quadratic in time derivatives of the field theory action. Linear terms in time derivatives of the fields lead to a linear term in the velocity in the mechanical Lagrangian, inducing a Lorentz force. Finally, the static part of the field theoretical energy produces the mechanical potential energy. With this procedure, the adiabatic method fixes the geometric structure of the moduli space $\mathcal{M}_n$: the zeroth, first- and second-order terms in time derivatives entering in the field theory action respectively supply the definition of the manifold $\mathcal{M}_n$ itself, its complex structure, and its metric. This way of proceeding has been successfully applied in a variety of models; there are, for example, some works on the AH model, both at the self-dual point [13,14] or away from it [15], that lead to second-order dynamics without a Lorentz term, but Manton has also shown [16] how the same vortices can be embodied in a theory that has purely first-order dynamics.

The adiabatic method has also been applied to the analysis of the scattering of CSH vortices in [17]. In this system, the approach runs into difficulties, and the reasons why the method fails are pointed out in the same work. Regarding this problem, we realize that the same moduli space of self-dual vortices can be part of different field theories, as Manton discovered for the Nielsen–Olesen vortices at critical coupling. We shall therefore start from a fixed moduli space, $\mathcal{M}_n$, of topological vortices and search for "simple" Lagrangians such that the points of $\mathcal{M}_n$ will be absolute minima of the field energy. Here we take the simplicity requirement as having the most natural dynamics, i.e., that associated with Manton's approach. We shall see that the simultaneous existence of first- and second-order time-derivative terms, as it occurs in CSH models, leads to a complex dynamical system on the moduli space, and that difficulties appear in a complete analytical treatment. In this paper, however, we shall study two generalized Abelian Higgs models that share the same moduli space of vortices with the CSH system. The first of the models is non-relativistic, and first-order vortex dynamics arises, captured at the adiabatic limit. The other model is relativistic, and the vortices evolve according to second-order dynamics. Comparison with the application of the adiabatic method to the CSH system helps to clarify the origin of the problems found in this model. In our analysis we find a universal kind of behaviour: The low-energy dynamics of topological CSH vortices in the non-relativistic model resembles the adiabatic limit of the Ginzburg–Landau theory proposed in [16]. Topological vortices in the generalized AH model scatter at low energies as do Nielsen–Olesen vortices in the AH model. When first- and second-order dynamics are entangled, the adiabatic limit becomes very cumbersome. The rest of the paper is organized as follows. In the next section, the CSH vortices are introduced, the issue of low-energy dynamics is addressed and its difficulties made clear. This is based on previous work performed in [18] and [17]. The next two sections are devoted to studying two alternative dynamics for the same vortices: new first- and second-order vortex dynamics, as arise in a non-relativistic model and a relativistic one, respectively, are discussed. Some further comments and brief general conclusions are offered in the last section.

2 The adiabatic limit and CSH vortex dynamics

2.1 The moduli space of vortices in the CSH model

The action of the Abelian Chern–Simons–Higgs gauge system is [7,8]

$$S = \int \mathrm{d}^3x \left\{ \frac{\kappa}{4}\varepsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + \frac{1}{2}D_\mu\phi^* D^\mu\phi - \frac{\lambda}{8}|\phi|^2(|\phi|^2 - v^2)^2 \right\} \quad (1)$$

where the space-time is three-dimensional, the metric is $g_{\mu\nu} = \mathrm{diag}(1,-1,-1)$, and the covariant derivative is $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$. The Lagrangian is quasi-invariant against the gauge transformations

$$\phi \to \mathrm{e}^{ie\Lambda}\phi, \qquad A_\mu \to A_\mu - \partial_\mu\Lambda. \quad (2)$$

Bearing in mind that

$$\frac{\kappa}{4}\varepsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} = \frac{\kappa}{2}\varepsilon_{kl}\dot{A}_k A_l + \kappa A_0 F_{12} + \text{divergence},$$
$$D_0\phi^* D^0\phi = (\partial_0|\phi|)^2 + |\phi|^2(\mathrm{e}A_0 + \partial_0\arg(\phi)), \quad (3)$$

and eliminating A_0 by means of the Gauss law coming from (1),

$$A_0 = -\frac{\kappa F_{12}}{|\phi|^2} - \frac{1}{e}\partial_0\arg(\phi), \quad (4)$$

the action separates into kinetic and potential parts as follows:

$$S = \int \mathrm{d}t\{T - V\} \quad (5)$$

$$T = \int \mathrm{d}^2x \left\{ \frac{1}{2}\dot{\varphi}^2 + \frac{\kappa}{2}\varepsilon_{kl}\dot{A}_k A_l - \frac{\kappa}{2e}\dot{\Theta}F_{12} \right\} \quad (6)$$

$$V = \int \mathrm{d}^2x \left\{ \frac{1}{2}\frac{\kappa^2}{e\varphi^2}F_{12}^2 + \frac{1}{2}D_k\phi^* D_k\phi + \frac{\lambda}{8}\varphi^2(\varphi^2-1)^2 \right\} \quad (7)$$

where $\phi = \varphi\mathrm{e}^{i\Theta/2}$. Passing to the Hamiltonian formalism, we find

$$H = \int \mathrm{d}t\{K + V\} \quad (8)$$

with

$$K = \frac{1}{2}\int \mathrm{d}^2x\dot{\varphi}^2. \quad (9)$$

We shall first focus on static configurations. For these, $L = -V$, $H = V$ and the finiteness of the energy requires $\phi(\vec{x}) \to 0$ or v when $|\vec{x}| \to \infty$. In this paper, we limit ourselves to the second case, i.e., we will work on the configuration space

$$\mathcal{C} = \{\Gamma \equiv (\phi, A_k)/\dot{\Gamma} = 0, E[\Gamma] < \infty, \phi|_{|\vec{x}|\to\infty} = v\}. \quad (10)$$

Each configuration in $\mathcal{C}$ gives rise to a map from the boundary of the plane at infinity to the gauge group, $\phi_\infty : S^1_\infty \to U(1)$ and is therefore associated with an integer, the winding number n of ϕ_∞. As a consequence, $\mathcal{C} = \cup_{n\in\mathbf{Z}}\mathcal{C}_n$ and a topological superselection rule arises: Time evolution cannot change the initial winding number. Furthermore, because $D_k\phi$ must vanish at infinity, the magnetic flux of the configurations in $\mathcal{C}_n$ is given by $\Phi_\mathrm{M} \equiv -\int \mathrm{d}^2x F_{12} = 2\pi n/e$.

Our interest lies in the solutions belonging to $\mathcal{C}_n$, which are topological n-vortices. Although the theory also includes another class of very interesting non-topological solutions with a vanishing asymptotic scalar field, there is evidence that such non-topological solutions can be understood as assemblies of vortices mixed with some basic non-topological defects [19]. Hence, the dynamics of this kind of solution only differs from that of the topological vortices in the effect of the vortex-defect interaction, an issue to be dealt with elsewhere. In order to render V extremal in $\mathcal{C}$, the Bogomolnyi trick is useful:

$$V = \int \mathrm{d}^2x \left\{ \frac{1}{2}\left[\frac{\kappa F_{12}}{\mathrm{e}\varphi} \mp \frac{e^2}{2\kappa}\varphi(\varphi^2 - v^2)\right]^2 \right.$$

$$+\frac{1}{2}|D_1\phi \pm iD_2\phi|^2$$
$$+\frac{1}{8}\left(\lambda - \frac{e^4}{\kappa^2}\right)\varphi^2(\varphi^2 - v^2)^2\Bigg\} \pm \frac{ev^2}{2}\Phi_M. \quad (11)$$

There is a critical point at $\lambda = e^4/\kappa^2$ where the contribution of the third term vanishes and a global lower bound to the energy arises: $V \geq \pi v^2|n|$ for any configuration in $\mathcal{C}_n$. The bound is saturated if and only if the first order equations

$$eF_{12} = \pm\frac{m^2}{2}\frac{\varphi^2}{v^2}\left(\frac{\varphi^2}{v^2} - 1\right) \quad (12)$$
$$D_1\phi \pm iD_2\phi = 0, \quad (13)$$

where $m = e^2v^2/\kappa$, are satisfied; solutions of (12, 13) are also solutions of the Euler–Lagrange equations. We see that by replacing the Maxwell term with the Chern–Simons term, self-duality requires a potential of sixth order in the modulus of the Higgs field. Below we fix the upper sign in these equations and work on $\mathcal{C}_n$ with $n > 0$; the opposite choice would lead to analogous antivortices with $n < 0$.

Using the Poincaré $\bar{\partial}$ lemma, it is possible to prove that the Higgs field of the non-singular solutions of (13) has exactly n zeros, and that away from them, the phase Θ is regular [20,21]. Furthermore, near a zero $\vec{q}$ of order r, the field behaviour is

$$\varphi \simeq c|\vec{x} - \vec{q}|^r \qquad \Theta \simeq 2r\theta(\vec{x} - \vec{q}) \quad (14)$$

$\theta(\vec{x})$ being the polar angle of $\vec{x}$. The self-duality equations over $\mathbf{R}^2 - \{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n\}$ are

$$\nabla^2 u = m^2 \mathrm{e}^u(\mathrm{e}^u - 1) \quad (15)$$
$$eA_k = -\frac{1}{2}(\partial_k\Theta + \varepsilon_{kj}\partial_j u) \quad (16)$$

with $u = \ln(\varphi/v)^2$. Observe that with respect to the corresponding equations in the AH model, there is an additional factor, e^u, on the right-hand side of equation (15). The manifold of solutions of (12) and (13) on $\mathcal{C}_n$, modulo the group of gauge diffeomorphisms, is the n-vortex moduli space $\mathcal{M}_n$. As proven by Wang [20], $\mathcal{M}_n$ is the smooth manifold of unordered n-points in $\mathbf{C}$: $\mathcal{M}_n = \mathbf{C}^n/\Sigma^n$, where Σ^n is the symmetric group of $n!$ elements. This is so because the n zeros of ϕ in $\mathbf{C} \simeq \mathbf{R}^2$ determine a unique solution, up to permutation and gauge equivalence. A system of "good" coordinates in $\mathcal{M}_n$ is provided by the coefficients of the complex monic polynomial of degree n whose roots are the zeros of ϕ: $P(z) = z + a_1 z^{n-1} + ... + a_n$, with $P(z_a) = \phi(z_a) = 0$ for $z_a = q_a^1 + \mathrm{i}q_a^2$, $a = 1, 2, ..., n$. Had we chosen the centres of the vortices z_a as a system of coordinates in $\mathcal{M}_n$, singularities would have appeared when two zeros coincided. From a physical viewpoint, the structure of $\mathcal{M}_n$ shows that at the self-dual limit, the scalar attractive force and the gauge repulsive force compensate each other mutually, hence the static self-dual solutions consist of systems of non-interacting vortices.

2.2 The dynamics of slowly moving vortices

We now address the issue of the time evolution of a self-dual system of vorticity n. Because the time-dependent field equations are too difficult to solve, it is necessary to restrict the problem in such a way that an approximate treatment is feasible. The most natural restriction is to limit ourselves to the case of very slowly evolving fields so that we can address the problem with Manton's adiabatic method: The point is that the solutions $\Gamma[\vec{x}, t]$ with $\dot{\Gamma}$ small essentially describe the motion of the individual vortices. We can thus identify $\Gamma[\vec{x}, t]$ with a curve $\{\vec{q}_a(t)\}$ in $\mathcal{M}_n$, i.e.,

$$\Gamma(\vec{x}; t) = \Gamma(\vec{x}; \vec{q}_a(t)), \qquad \dot{\Gamma}(\vec{x}; t) = \frac{\partial\Gamma(\vec{x}; \vec{q}_a(t))}{\partial q_a^k}\dot{q}_a^k. \quad (17)$$

The field-theoretical problem is transmuted to a $2n$-dimensional mechanical one; introduction of (17) into (5) and integration to the whole plane afford a Lagrangian $L = T(\vec{q}_a, \dot{\vec{q}}_a) - V(\vec{q}_a)$ whose variational equations admit as a solution the curve $\{\vec{q}_a(t)\}$ in $\mathcal{M}_n$ that corresponds to some given initial conditions. In fact, on the moduli space, $V(\vec{q}_a) = \pi n v^2$ and the only important term of L is the kinetic one.

To carry out this program, the first step is to unequivocally fix the form of $\Gamma[\vec{x}; \vec{q}_a]$ i.e, to fix the gauge by defining $\Theta(\vec{x}; \vec{q}_a)$ locally on the moduli. This gauge fixing must be done in such a way that the kinetic energy will be invariant, not only against the group $\mathcal{G}$ of gauge diffeomorphisms but also against the enlarged group $\tilde{\mathcal{G}}$ of moduli-dependent gauge transformations; the dynamics cannot vary if we choose different gauges in different points of the moduli. However, despite this strong requirement, in the CSH model there is no restriction on our freedom to choose $\Theta(\vec{x}; \vec{q}_a)$; the only requisite is to respect the boundary conditions in $\{\vec{x} = \vec{q}_a\}$ and S^1_∞. The reason for this is that (6) is invariant against moduli-dependent gauge transformations because this expression was obtained from formula (1) merely by imposing the gauge-independent Gauss law (4). Hence, we can fix the gauge in the simplest form compatible with the boundary conditions

$$\Theta(\vec{x}; \vec{q}_a) = 2\sum_{a=1}^{n}\theta(\vec{x} - \vec{q}_a), \quad (18)$$

i.e, by extending the known behaviour near the centers of the vortices to the whole plane. Introduction of (18) into (16) gives

$$\mathrm{e}A_k(\vec{x}; \vec{q}_a) = \varepsilon_{kj}\partial_j\xi(\vec{x}; \vec{q}_a) \quad (19)$$

where

$$\xi(\vec{x}; \vec{q}_a) = -\frac{1}{2}u(\vec{x}; \vec{q}_a) + \sum_{a=1}^{n}\ln|\vec{x} - \vec{q}_a|; \quad (20)$$

Therefore, ξ is regular on the whole $\mathbf{R}^2$, see (14). Using (19) and computing its time derivative, we obtain

$$\varepsilon_{kl}\dot{A}_k A_l = \partial_j(\dot{\xi}A_j) - \dot{\xi}(\partial_j A_j) \quad (21)$$

Because the vector field of the vortices is transverse, the second term in (6) is a global divergence and can be dropped. The third term in the kinetic energy can be written in the form

$$\frac{1}{2}\dot{\Theta}F_{12} = \varepsilon_{ij}\varepsilon_{kl}\partial_k A_l \sum_{b=1}^{n} \dot{q}_b^i \partial_j \ln|\vec{x}-\vec{q}_b|, \quad (22)$$

using (18). This expression is regular across the entire plane because F_{12} vanishes at the center of the vortices. Proceeding by partial differentials, one can see that, besides an irrelevant divergence,

$$\frac{1}{2}\dot{\Theta}F_{12} = 2\pi \sum_{b=1}^{n} A_i(\vec{x})\dot{q}_b^i \delta(\vec{x}-\vec{q}_b)\ . \quad (23)$$

Following [14], we expand the modulus of the Higgs field near the bth vortex, in the form

$$\frac{1}{2}u(\vec{x};\vec{q}_a)|_{\vec{x}\simeq\vec{q}_b} = \ln|\vec{x}-\vec{q}_b| + a_b + \vec{b}_b\cdot(\vec{x}-\vec{q}_b) + \ldots; \quad (24)$$

where $a_b, \vec{b}_b$ are functions of the $\vec{q}$; then the value of the vector field at the center of that vortex is

$$eA_k(\vec{q}_b;\vec{q}_a) = \varepsilon_{kj}\left[\sum_{a\neq b}\frac{q_b^j - q_a^j}{|\vec{q}_b-\vec{q}_a|^2} - b_b^j\right] \quad (25)$$

for any solution of the vortex equations.

Substitution of (25) into formula (23) produces a term in the kinetic energy (6) that involves only the $\vec{q}$ and their time derivatives. Unfortunately, it is not possible to obtain an explicit expression for the remaining quadratic term in closed form, because integration to the whole plane requires detailed knowledge of $\varphi(\vec{x};\vec{q}_a)$ as a function of $\vec{x}$, and not only in the vicinity of each vortex. Because the exact solution of the system in (12) and (13) is unknown, the best thing that we can do is to write the mechanical Lagrangian in the form

$$L = \frac{1}{2}\sum_{a,b=1}^{n} g_{\rm ij}^{ab}\dot{q}_a^{\rm i}\dot{q}_b^j - \frac{2\pi\kappa}{\rm e}\sum_{b=1}^{n}\dot{q}_b^k A_k(\vec{q}_b;\vec{q}_a) - \pi v^2 n \quad (26)$$

with A_k given by (25) and

$$g_{ij}^{ab} = \int {\rm d}^2x \frac{\partial\varphi}{\partial q_a^i}\frac{\partial\varphi}{\partial q_b^j}. \quad (27)$$

The only possibility for integrating (27) to obtain an analytic expression for the metric is to consider the asymptotic regimes of either very close or very separated vortices. We now analyze the second case, in which the scalar field around each vortex is approximately radially symmetric, i.e., the $\vec{b}$ are vanishingly small. This behaviour and the great distance among vortices guarantee that the vector field at the centers is negligible (see (25)), and that the dynamics is governed by the quadratic term in T. Because φ tends to v exponentially when $|\vec{x}-\vec{q}|$ goes to infinity, it makes sense to write

$$\varphi(\vec{x}) = \begin{cases} \varphi_1(|\vec{y}_a|) & \text{if } |\vec{y}_a| < R_v \\ v & \text{if } |\vec{y}_a| > R_v \end{cases} \quad (28)$$

where $\vec{y}_a = \vec{x}-\vec{q}_a$, φ_1 is the magnitude of the Higgs field of the radially symmetric 1-vortex, and R_v its characteristic radius, i.e., the radius of the circle in which φ_1 differs appreciably from v. It is then easy to see that

$$g_{ij}^{ab} = \delta^{ab}\int {\rm d}^2y \frac{y^i y^j}{r^2}\left(\frac{{\rm d}\varphi_1}{{\rm d}r}\right)^2, \quad (29)$$

with $r = |\vec{y}|$, or

$$g_{ij}^{ab} = \delta^{ab}\delta_{ij}M, \qquad M = \frac{1}{2}\int {\rm d}^2y\left(\frac{{\rm d}\varphi_1}{{\rm d}r}\right)^2. \quad (30)$$

Plugging the radial form of equation (13) into this expression,

$$\frac{{\rm d}\varphi_1}{{\rm d}r} = \frac{1+eA_\theta}{r}\varphi_1, \quad (31)$$

we find

$$M = -\frac{e}{4}\int {\rm d}^2x\varphi_1^2 F_{12}. \quad (32)$$

But $\varphi_1^2 < v^2$, so we conclude that $M < \pi v^2/2$. This is an inconsistent answer, implying that the inertia of each vortex is less than half its mass, which for the case $n = 1$ leads to a conflict with relativistic invariance. This strongly suggests that the adiabatic approach fails in the CSH model and needs to be improved. The critical analysis of the adiabatic method in the current model, carried out by Dziarmaga [17], reveals the reason for the failure. We review this analysis in the next subsection.

2.3 The improved adiabatic limit

Consider a general field theory with a field multiplet (ψ_a) and a Lagrangian

$$\mathcal{L} = G_{ab}[\psi]\dot{\psi}_a\dot{\psi}_b + K_a[\psi]\dot{\psi}_a - \mathcal{H}[\psi] \quad (33)$$

where there are no time derivatives inside the brackets. Assume that the static solutions of the field equations form a moduli space $\mathcal{M}$. $V = \int {\rm d}^n x\mathcal{H}[\psi]$ takes the same constant value on each point of $\mathcal{M}$. Let $\{\lambda_A\}$ be a local system of coordinates in $\mathcal{M}$ and let $\varphi_a(\vec{x};\lambda_A)$ denote the fields corresponding to the solution $\{\lambda_A\}$. At Manton's adiabatic limit, slow time evolution merely amounts to motion in the moduli space. Thus, time dependence is due exclusively to variations in the $\{\lambda_A\}$ coordinates as functions of time,

$$\dot{\psi}_a = \frac{\partial\varphi_a}{\partial\lambda_A}\dot{\lambda}_A, \quad (34)$$

hence, the effective Lagrangian

$$\mathcal{L}_{\rm eff}^{\rm Manton} = G_{ab}[\varphi]\frac{\partial\varphi_a}{\partial\lambda_A}\frac{\partial\varphi_b}{\partial\lambda_B}\dot{\lambda}_A\dot{\lambda}_B + K_a[\varphi]\frac{\partial\varphi_a}{\partial\lambda_A}\dot{\lambda}_A - \mathcal{H}[\varphi]. \quad (35)$$

is obtained. However, the true solutions of the time-dependent Euler–Lagrange equations are configurations $\psi_a(\vec{x},t) \neq \varphi_a(\vec{x};\lambda_A(t))$. In principle, one could improve the adiabatic approach, even without knowledge of the exact solutions of the time-dependent non-linear field equations, by the inclusion of a linear term in $\dot{\lambda}_A$,

$$\psi_a(\vec{x},t) = \varphi_a(\vec{x};\lambda_A(t)) + \phi_a^B(\vec{x};\lambda_A(t))\dot{\lambda}_B(t), \tag{36}$$

that accounts for the deformation of the static fields as a result of the motion. (34) now becomes

$$\dot{\psi}_a(\vec{x},t) = \frac{\partial\varphi_a}{\partial\lambda_A}\dot{\lambda}_A + \phi_a^A\ddot{\lambda}_A + \frac{\partial\phi_a^B}{\partial\lambda_A}\dot{\lambda}_A\dot{\lambda}_B, \tag{37}$$

and the introduction of (36) and (37) into (33) gives a very complicated expression,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{(2)} + \mathcal{L}_{\text{eff}}^{(1)}, \tag{38}$$

where

$$\mathcal{L}_{\text{eff}}^{(1)} = K_a[\varphi]\dot{\varphi}_a, \quad \text{and} \tag{39}$$

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{(2)} = {} & G_{ab}[\varphi](\dot{\varphi}_a\dot{\varphi}_b + 2\dot{\varphi}_a\dot{\Delta}_b + \dot{\Delta}_a\dot{\Delta}_b)\\ & + \frac{\delta K_a}{\delta\psi_b}(\dot{\varphi}_a\Delta_b - \dot{\varphi}_b\Delta_a - \dot{\Delta}_b\Delta_a)\\ & - \frac{1}{2}\frac{\delta^2\mathcal{H}}{\delta\psi_a\delta\psi_b}\Delta_a\Delta_b,\end{aligned} \tag{40}$$

with

$$\Delta_a = \phi_a^B\dot{\lambda}_B, \qquad \dot{\Delta}_a = \phi_a^A\ddot{\lambda}_A + \frac{\partial\phi_a^B}{\partial\lambda_A}\dot{\lambda}_A\dot{\lambda}_B. \tag{41}$$

The question is whether or not this modification, which we shall call the improved adiabatic limit, has any physical meaning. There are three different cases:

A. Assume that both G_{ab} and K_a are different from zero. Because $\dot{\varphi}_a$ is linear in λ_A, $\mathcal{L}_{\text{eff}}^{(1)}$ is linear in the velocities and all the terms entering in $\mathcal{L}_{\text{eff}}^{(2)}$ are at least quadratic. Integration over the whole plane of $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{(2)} + \mathcal{L}_{\text{eff}}^{(1)}$ leads to a reduced mechanical Lagrangian taking the form

$$L_{\text{eff}} = g_{AB}(\lambda)\dot{\lambda}_A\dot{\lambda}_B + h_A(\lambda)\dot{\lambda}_A, \tag{42}$$

which describes the motion on the moduli space $\mathcal{M}$. Because the energy is constant on $\mathcal{M}$, the static forces between vortices are null, i.e., $\ddot{\lambda}_A = \omega_{AB}(\lambda)\dot{\lambda}_B + o(\dot{\lambda}^2)$; the acceleration is zero if the velocity is zero. Because of the linear term in L_{eff}, $\omega_{AB}(\lambda)$ is not zero; hence Δ_a and $\dot{\Delta}_a$ are of the same order. Therefore, the quadratic term in the velocities in $\mathcal{L}_{\text{eff}}^{\text{Manton}}$ should be replaced by $\mathcal{L}_{\text{eff}}^{(2)}$, hence one should consider the improved adiabatic limit. This is exactly the case in the CSH system:

$$G_{ab}\dot{\psi}_a\dot{\psi}_b = \frac{1}{2}\dot{\varphi}^2, \qquad K_a[\psi]\dot{\psi}_a = \frac{\kappa}{2}\varepsilon_{kl}\dot{A}_kA_l - \frac{\kappa}{2e}\dot{\Theta}F_{12}. \tag{43}$$

The dynamics of the CSH vortices at the improved adiabatic limit is governed by the mechanical Lagrangian L_{eff}, where g_{AB} and h_A are derived from $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{(2)} + \mathcal{L}_{\text{eff}}^{(1)}$. This is a very difficult problem; first, it is not possible to give a closed expression for the metric $g_{AB}(\lambda)$ because it depends not only on the vortex motion in the moduli space, but also includes effects coming from the field deformations whose specifications are beyond self-duality, and this makes the use of the Euler-Lagrange equations unavoidable. Moreover, there are Lorentz forces due to $h_A(\lambda)$ that would strongly disturb the possible geodesic motion in the metric g_{AB}.

B. Let us next consider the case where G_{ab} is not zero, but K_a is zero. The mechanical Lagrangian is now

$$L_{\text{eff}} = g_{AB}(\lambda)\dot{\lambda}_A\dot{\lambda}_B \tag{44}$$

and $\omega_{AB}(\lambda) = 0$. Then, $\ddot{\lambda}_A = -\Gamma_{ABC}(\lambda)\dot{\lambda}_B\dot{\lambda}_C$ and $\mathcal{L}_{\text{eff}}^{(2)} = \mathcal{L}_{\text{eff}}^{\text{Manton}}$, up to quadratic order in the velocities. The remaining terms in $\mathcal{L}_{\text{eff}}^{(2)}$ are at least cubic in $\dot{\lambda}_A$. This statement is obvious for the terms in the first line of formula (40), but we now need the field equations derived from (33) to check that $\delta^2\mathcal{H}/\delta\psi_a\delta\psi_b$ is indeed proportional to $\dot{\lambda}_A$ at slow velocities. Thus, the modification induced by formula (36) is negligible, and the adiabatic limit is tantamount to geodesic motion in the moduli space. The Abelian Higgs model obeys this situation, with

$$G_{ab}\dot{\psi}_a\dot{\psi}_b = \frac{1}{2}(\dot{\phi}^*\dot{\phi} + \dot{A}_i\dot{A}_i), \tag{45}$$

and the low-energy dynamics of vortices becomes a workable mechanical problem. In Sect. 4 we shall discuss a generalized Abelian Higgs model that also belongs to this type.

C. Finally, let us consider the opposite situation: G_{ab} is zero, but K_a is not null. The key point is that, in this case, $\mathcal{L}_{\text{eff}}^{\text{Manton}}$ is linear in velocities and we do not need to consider the corrections induced in $\mathcal{L}_{\text{eff}}^{(2)}$ by deformations of the fields, because they are at least of second order in $\dot{\lambda}_A$. The low-energy dynamics is again captured by the adiabatic limit, which now consists of a mechanical problem on the configuration space $\mathcal{M}$ with Lagrangian

$$L_{\text{eff}} = h_A[\lambda]\dot{\lambda}_A, \tag{46}$$

causing motion on the moduli space that is due exclusively to Lorentz forces. The non-relativistic Ginzburg-Landau system analyzed in [16] belongs to this type:

$$K_a[\psi]\dot{\psi}_a = \mathrm{i}(\phi^*\dot{\phi} - \phi\dot{\phi}^*) + \frac{\kappa}{2}\varepsilon_{kl}\dot{A}_kA_l - \frac{\kappa}{2\mathrm{e}}\dot{\Theta}F_{12}. \tag{47}$$

A generalization of this model in the same class will be studied in the next section.

The conclusion is that the usual adiabatic approach is suitable for studying slow motion dynamics when the system is purely linear or quadratic in the time derivatives of the fields, but not when there are terms of both types simultaneously. In this case, the approach needs to be refined, and this leads to an exceedingly complicated problem that does not admit any analytical treatment [17].

* * *

To close this section we briefly discuss the issue of vortex CSH statistics, a rather paradoxical subject [18]. For a topological CSH vortex at rest, we have:

$$\Phi_{\mathrm{M}} = \frac{2\pi}{e}, \qquad Q = -\frac{2\pi\kappa}{e}, \qquad J = -\frac{\pi\kappa}{e^2}. \tag{48}$$

If we trust the standard computation of the statistical angle of two-dimensional anyons through the Aharonov–Bohm (AB) effect, we find that the CSH vortices correspond to a statistic $\nu = 2\pi\kappa/\mathrm{e}^2$ and the spin-statistics relation is $\nu = -2s$; there is a minus sign with respect to the expected outcome. Nevertheless, in the adiabatic mechanical Lagrangian (26), the term

$$L_{\mathrm{stat}} = -\frac{2\pi\kappa}{e^2}\varepsilon_{kj}\sum_b \dot{q}_b^k \sum_{a<b}\frac{q_b^j - q_a^j}{|\vec{q}_a - \vec{q}_b|^2} \tag{49}$$

$$= +\frac{2\pi\kappa}{e^2}\sum_{a<b}\frac{\mathrm{d}}{\mathrm{d}t}\arg(\vec{q}_a - \vec{q}_b) \tag{50}$$

should be interpreted as providing anyonic statistics for a statistical angle $\nu = -2\pi\kappa/e^2 = 2s$, see [22]. We find the right answer at the adiabatic limit, whereas application of the AB method to extended distributions of electric and magnetic charge fails.

3 The non-relativistic linear model

The paradigm of linear gauge theory in the time derivatives is the non-relativistic model of Jackiw and Pi [23], which describes the minimal coupling between the non-linear Schrödinger matter field and the Chern–Simons gauge field in $(2+1)$ dimensions. Although this model contains self-dual vortices, these are quite different from that considered in the previous section. In the Jackiw–Pi (JP) theory one has only the symmetric phase, constructed on the unique vacuum $\phi = 0$, and the vortices are non-topological even if the magnetic flux is an integer. In fact, the JP model is the non-relativistic limit of the CSH system, and the JP vortices are the corresponding limit of the non-topological CSH vortices; the topological vortices disappear from the spectrum in the non-relativistic regime, and the flux quantization is due to the inversion properties of the Liouville equation rather than to topological reasons (the JP model enjoys conformal invariance and the vortex equations become equivalent to the Liouville equation). As we shall see, to have true self-dual topological vortices, the original JP theory must be modified with care.

3.1 The generalized Ginzburg–Landau theory

We shall now discuss a non-relativistic model with both symmetric and asymmetric phases. The new system is a generalization of the model analyzed by Manton in [12]; the crucial difference is that the moduli space of topological vortices is now the same as in the CSH theory, instead of being the moduli space of Ginzburg–Landau vortices. Of course, we shall find first-order vortex dynamics rather than the awkward situation of the CSH model. Before this, however, we must deal with the tricky question of making non-relativistic dynamics compatible with the spontaneous symmetry breakdown of U(1) invariance. Even though it is possible to build a non-symmetric vacuum in the JP theory, the fields cannot reach it asymptotically because that would lead to pathologies, namely, infinite charges and a misdefinition of the canonical formalism. Barashenkov and Harin [24] traced the origin of the problem back to the underlying pure scalar model in 1+1 dimensions and found that a possible loophole is to multiply the $\dot{\phi}$ term of the Lagrangian by a factor of $1 - (|\phi|^2/v^2)$. To determine this factor, they used the condition that the Euler–Lagrange equations of the modified scalar model must coincide with those of the original one. However, since the theory is to be gauged, this is perhaps too restictive a requirement. Instead, one can consider a more general version of the non-linear Schrödinger Lagrangian in 1+1 dimensions of the form

$$\mathcal{L} = \frac{\mathrm{i}}{2}H(\varphi)[\phi^*\partial_0\phi - \phi\partial_0\phi^*] - \partial_x\phi^*\partial_x\phi - U(\varphi) \tag{51}$$

where U is a potential that includes an asymmetric vacuum of modulus v. From (51) we obtain the conserved current

$$\rho = \varphi^2 H(\varphi), \qquad j_x = -\mathrm{i}(\phi^*\partial_x\phi - \phi\partial_x\phi^*), \tag{52}$$

and the field momentum

$$P = \frac{\mathrm{i}}{2}\int \mathrm{d}x H(\varphi)[\phi^*\partial_x\phi - \phi\partial_x\phi^*] \tag{53}$$

whose variation is given by

$$\begin{aligned}\delta P = \mathrm{i}\int \mathrm{d}x\{&[H\partial_x\phi - \partial_x(H\phi)]\delta\phi^* \\ &-[H\partial_x\phi^* - \partial_x(H\phi^*)]\delta\phi\} \\ &+\frac{\mathrm{i}}{2}\int \mathrm{d}x\{\partial_x[H\phi^*\delta\phi - H\phi\delta\phi^*] \\ &+\frac{\mathrm{d}H}{\mathrm{d}\varphi^2}[\phi\delta\phi^* + \phi^*\delta\phi]\}.\end{aligned} \tag{54}$$

The difficulties emphasized in [24] are twofold: If $\phi(\pm\infty) = v\mathrm{e}^{i\chi_\pm}$ and $H(\varphi) = 1$, the charge $Q = \int \mathrm{d}x\rho$ diverges, and the momentum variation includes a term $v^2[\delta\chi_+ - \delta\chi_-]$ that cannot be differentiated with respect to $\delta\phi$ or $\delta\phi^*$, and therefore the canonical formalism is perturbed. However, it suffices to introduce any $H(\varphi)$ such that $H(v) = 0$ to avoid both problems: δP will then be well defined, and Q will be finite and will vanish for all vacua.

We now turn to the gauge theory and propose the following modified Jackiw–Pi model

$$S = \int \mathrm{d}^3x \Big\{ \frac{\mathrm{i}}{2} H(\varphi)[\phi^* D_0\phi - \phi D_0\phi^*] + \frac{\kappa}{4}\varepsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} - \frac{1}{4}G(\varphi)F_{ij}F_{ij} - \frac{1}{2}D_k\phi^* D_k\phi - \frac{\lambda\kappa^2}{8e^2 G(\varphi)}(\varphi^2 - v^2)^2 \Big\} \quad (55)$$

where we follow Manton's clever idea [16] of taking advantage of the Galilean invariance in order to include an asymmetric Maxwell term without any contribution from the electric field. Nevertheless, we avoid use of external couplings to maintain the gauge invariance of the theory explicitly. Furthermore, we use a dielectric function $G(\varphi)$ to build up a non-minimal interaction between the scalar and gauge fields, as is done in [25]. Below, we shall treat this issue more generally, but for the time being, we set

$$G(\varphi) = \frac{\kappa^2}{e^2\varphi^2}. \quad (56)$$

Apart from the above motives, there is another reason for including the function $H(\varphi)$ in the Lagrangian. It has recently been shown in [26] that the effective theory for the low-energy interaction between a planar relativistic fermionic gas and a crystalline background leads to a Chern–Simons–Higgs model in which the term in the covariant derivatives is multiplied by a function H; this function is fixed by self-duality and supersymmetry criteria. In a non-relativistic situation, it is not necessary to use the same function for temporal and spatial derivatives, so in (55) we have chosen $H = 1$ for the spatial derivatives. As we shall see, in a vorticial arena, $|H| < 1$, so including this function as a factor in only the temporal term favours a small deformation of the vortices in their low-energy motion.

Among the Euler–Lagrange equations from (55) we find the Gauss law

$$\kappa F_{12} - e\varphi^2 H(\varphi) = 0 \quad (57)$$

whose form is exactly as in (4), $\kappa F_{12} - e\rho = 0$; therefore (55) and (1) give rise to the same anyonic statistics [22]. To split (55) into kinetic and potential parts, it is convenient to adopt the temporal gauge $A_0 = 0$. Then

$$S = \int \mathrm{d}t(T - V), \quad (58)$$

$$T = \int \mathrm{d}^2x \left\{ \frac{\mathrm{i}}{2} H(\varphi)(\phi^*\dot\phi - \phi\dot\phi^*) + \frac{\kappa}{2}\varepsilon_{kl}\dot A_k A_l \right\}, \quad (59)$$

and V coincides with (7). Because A_0 is not present in (58), the Gauss law (57) must be imposed as a constraint on the field equations arising from (58).

We now turn to studying the static limit of the theory. Given that V is the same as in the CSH model, the whole analysis of the static part of that model is still valid in our non-relativistic theory: At the self-dual limit $\lambda = \mathrm{e}^4/\kappa^2$, the solutions of equations (12) and (13) are extrema of the action (55), and in the $\mathcal{C}_n$ sector, have $V = \pi v^2 n$, and form the moduli space $\mathcal{M}_n$ with local coordinates $\{\vec q_a\}$ corresponding to the positions of the zeros of ϕ. In this system too, the self-dual solutions are absolute minima of V. The specific forms of $U(\varphi)$ and $G(\varphi)$ force the function $H(\varphi)$ to be chosen as

$$H(\varphi) = \frac{\mathrm{e}^2}{2\kappa}(\varphi^2 - v^2) \quad (60)$$

in order to make the Gauss law (57) compatible with the vortex equations (12) and (13). It is remarkable that an identical choice of $H(\varphi)$ allows one to extend the generalization of the CSH system studied in [26] to an $N = 2$ SUSY theory.

3.2 First-order vortex dynamics

Introduction of $H(\varphi)$ into (59) and use of the Gauss law give

$$T = \int \mathrm{d}^2x \left\{ \frac{\kappa}{2}\varepsilon_{kl}\dot A_k A_l - \frac{\kappa}{2\mathrm{e}}\dot\Theta F_{12} \right\}, \quad (61)$$

i.e., on the moduli space, T is equal to the sum of the terms linear in time derivatives appearing in formula (6). This guarantees the gauge invariance of (61). Additionally, if we work in the gauge $\Theta(\vec x;\vec q_a) = 2\sum_{a=1}^n \theta(\vec x - \vec q_a)$ in particular, then the Manton approach, after the algebra already seen for the CSH model, leads to

$$L = -\frac{2\pi\kappa}{\mathrm{e}}\sum_{b=1}^n \dot q_b^k A_k(\vec q_b;\vec q_a) - V(\vec q_a) \quad (62)$$

where of course $A_k(\vec q_b;\vec q_a)$ is given by (25). Because (62) is linear in the time derivatives, the adiabatic limit is now completely satisfactory; there are no terms containing time derivatives in the field equations that become unimportant when time goes to ∞ at different rates (see [27] for a conceptual analysis of this situation). In order to obtain non-trivial dynamics, we must consider the "almost" self-dual regime [15,16], i.e., we take $\lambda = (\mathrm{e}^4/\kappa^2) + \mu$, $\mu \simeq 0$ and hence the vortices are subject to small static forces: In (62), $V(\vec q_a) = n\pi v^2 + \mu W(\vec q_a)$, where

$$W(\vec q_a) = \frac{1}{8}\int \mathrm{d}^2x \varphi^2(\vec x;\vec q_a)[\varphi^2(\vec x;\vec q_a) - v^2]^2. \quad (63)$$

Even though the precise expression of W cannot be found analytically, it is obvious from (11) that if $\mu > 0$, the energy of an assembly of several vortices increases, and if $\mu < 0$, it decreases; when this is compared with the self-dual case, it leads to the assumption that forces among vortices are repulsive for $\mu > 0$ (attractive for $\mu < 0$) and therefore that $W(\vec q_a)$ is smaller for larger intervortex distances, a conjecture that is supported by numerical computations [28] and theoretical arguments [21]. To appreciate the features of the dynamics derived from (62) it is convenient to analyze the $n = 2$ case in some detail. We fix the center of mass of the system to be the origin of coordinates, $\vec{Q} = (1/2)(\vec{q}_1+\vec{q}_2) = 0$, and work with the relative coordinate $\vec{q} = (1/2)(\vec{q}_1 - \vec{q}_2)$. Notice that $\vec{b}_1 = -\vec{b}_2$. To check this property, it suffices to recall that vortex indistinguishability requires that $\varphi(-\vec{q}+\vec{y};\vec{q}_a) = \varphi(\vec{q}-\vec{y};\vec{q}_a)$ and to use (24). Furthemore, because the system has to be symmetric with respect to parity transformations and rotations around the center of masses,

$$\vec{b}_1 = -\vec{b}_2 = \frac{1}{2}b(q)\vec{q} \tag{64}$$

Introducing (64) into (62), we find

$$L = -\frac{2\pi\kappa}{e^2}\left[\frac{1}{q^2} - b(q)\right]\varepsilon_{kj}\dot{q}^k q^j - \mu W(q). \tag{65}$$

In terms of the polar angle $\theta = \theta(\vec{q})$,

$$L = \frac{2\pi\kappa}{e^2}[1 - q^2 b(q)]\dot{\theta} - \mu W(q) \tag{66}$$

and the dynamical equations are

$$\begin{aligned} \dot{q} &= 0 \\ \dot{\theta} &= -\frac{\mu e^2}{2\pi\kappa}\frac{\mathrm{d}W/\mathrm{d}q}{2qb(q) + q^2\mathrm{d}b/\mathrm{d}q} \end{aligned} \tag{67}$$

($b(q)$ is not equal to $1/q^2$, as we will discuss later). Hence, the vortices move in circular orbits with constant angular speed. The magnitude of the angular speed is a function of the orbit radius, and the sense of movement is opposite for type I and type II superconductors.

To finish this subsection, we note that although we have fixed $G(\varphi)$ in (56) to make the moduli space of the model fit in with that of the CSH one, a similar treatment can be carried out for general $G(\varphi)$. The only difference is that expression (60) must be substituted with

$$H(\varphi) = \frac{\kappa}{2\varphi^2}\frac{\varphi^2 - v^2}{G(\varphi)}, \tag{68}$$

and that the self-duality equations are not (12) and (13) but rather a generalization that is to be addressed in the next section. Nevertheless, (61) and the subsequent results remain valid. All the vortices of the complete family of generalized non-relativistic models (55) have exactly the same first-order dynamics.

3.3 The effect of a charged background

All the generalized non-relativistic models governed by the action (55) can be modified by adding a charged constant background

$$S_B = e\int \mathrm{d}^3x v^2 A_0(\vec{x},t), \tag{69}$$

which leads to a new Gauss law

$$\kappa F_{12} = -e(v^2 + \varphi^2 H(\varphi)), \tag{70}$$

that renders the system self-dual at the static limit if $\lambda = e^4/\kappa^2$. The potential energy and Bogomolnyi equations in this system are (86) and (92, 93), respectively, as we shall see in the next section. The first-order equations for a general choice of $G(\varphi)$ are compatible with (70) if and only if H is chosen in the form:

$$H(\varphi) = \frac{v^2}{\varphi^2}\left[\frac{\kappa}{2G(\varphi)}\left(1 - \frac{\varphi^2}{v^2}\right) - 1\right]. \tag{71}$$

The price to be paid is a different choice of H. For instance, the model discussed by Manton in [16] corresponds to $G(\varphi) = \kappa/2$ and $H(\varphi) = -1$. The generalization of the system proposed in Sect. 3 obeys:

$$G(\varphi) = \frac{\kappa^2}{e^2\varphi^2}, \qquad H(\varphi) = \frac{e^2}{2\kappa}(v^2 - \varphi^2) - \frac{v^2}{\kappa^2}, \tag{72}$$

to be compared with formulas (56) and (57). Using the Gauss law (70) we see that the kinetic energy becomes:

$$T = \int \mathrm{d}^2x\left\{\frac{\kappa}{e}\dot{\Theta}F_{12} - \frac{\kappa}{2}\varepsilon_{kl}\dot{A}_k A_l + v^2\dot{\Theta}\right\}. \tag{73}$$

There is a new term with respect to the kinetic energy in the absence of the charged background, see (61), but before analyzing the physics arising from it, it is convenient to compare the developments of Sect. 3 with the parallel study in [16]. If we look at our choice of gauge $\Theta = 2\sum_{a=1}^n \theta(\vec{x} - \vec{q}_a)$ near the center of each vortex $\vec{x} = \vec{q}_b + \vec{\epsilon}$, and take the limit $|\vec{\epsilon}| \to 0$, we find:

$$\lim_{\epsilon\to 0}\Theta(\vec{x};\vec{\epsilon}) = 2\lim_{\epsilon\to 0}\sum_{a\neq b}\theta(\vec{q}_b + \vec{\epsilon} - \vec{q}_a) + 2\lim_{\epsilon\to 0}\theta(\vec{q}_b + \vec{\epsilon} - \vec{q}_b). \tag{74}$$

Solving the ambiguity by defining $\theta_b = \lim_{\epsilon\to 0}\theta(\vec{q}_b + \vec{\epsilon} - \vec{q}_b)$, we see that

$$\Theta(\vec{q}_b) = 2\sum_{a\neq b}\theta(\vec{q}_b - \vec{q}_a) + 2\theta_b = 2\psi_b + 2\theta_b, \tag{75}$$

which is exactly the Manton choice of gauge. To see how to glue these local choices, it suffices to look at the case of two vortices. Near the center of the first vortex we have:

$$\Theta(\vec{x})_{\vec{x}\to\vec{q}_1} \simeq \theta(\vec{x} - \vec{q}_1) + \psi_1, \qquad \Theta(\vec{q}_2) = \theta(\vec{q}_2 - \vec{q}_1) + \psi_1. \tag{76}$$

Around the second vortex, there are similar expressions:

$$\Theta(\vec{x})_{\vec{x}\to\vec{q}_2} \simeq \theta(\vec{x} - \vec{q}_2) + \psi_2, \qquad \Theta(\vec{q}_1) = \theta(\vec{q}_1 - \vec{q}_2) + \psi_2. \tag{77}$$

But in $\mathcal{M}_2$, the two descriptions above are equivalent: The impossibility of distinguishing the vortices requires $\Theta(\vec{q}_1) = \Theta(\vec{q}_2)$, and this identity determines the gluing by setting $\theta(\vec{q}_1 - \vec{q}_2) = \psi_2$ and $\theta(\vec{q}_2 - \vec{q}_1) = \psi_1$.

Consequently, we have found the same first-order dynamics as Manton, a result which is independent of the particular model under scrutiny. In systems that generalize the model analyzed in Sect. 3, the kinetic energy includes the first two terms of (73), and the reduced Lagrangian is:

$$L = -\frac{2\pi\kappa}{\mathrm{e}}\sum_{b=1}^{n}\varepsilon_{kj}\left[\sum_{a\neq b}\dot{q}_b^k\frac{q_b^j-q_a^j}{|\vec{q}_b-\vec{q}_a|^2}-\dot{q}_b^k b_b^j\right] = \frac{2\pi\kappa}{\mathrm{e}}\sum_{b=1}^{n}\frac{\mathrm{d}\psi_b}{\mathrm{d}t}+\frac{2\pi\kappa}{\mathrm{e}}\sum_{b=1}^{n}\varepsilon_{kj}b_b^j(q)\dot{q}_b^k. \quad (78)$$

The other contribution in (73), which is due to the existence of a constant background, leads to the reduced kinetic energy

$$T=\sum_{b=1}^{n}\int_{\Sigma}\mathrm{d}s\mathrm{d}t\hat{J}_{kj}^b[\gamma]\frac{\mathrm{d}q_b^k}{\mathrm{d}s}\wedge\frac{\mathrm{d}q_b^j}{\mathrm{d}t}=\sum_{b=1}^{n}\int_{\gamma=\partial\Sigma}\mathrm{d}t\hat{a}_j^b[\gamma]\frac{\mathrm{d}q_b^j}{\mathrm{d}t} \quad (79)$$

for a motion in the n-vortex moduli space along a closed path γ in $\mathcal{M}_n$. Here,

$$\hat{J}_{kj}^b[\gamma]=\frac{\partial\hat{a}_j^b}{\partial q_b^k}-\frac{\partial\hat{a}_k^b}{\partial q_b^j},\qquad \hat{a}_j^b(\vec{q}_b)=\varepsilon_{jk}q_k^b \quad (80)$$

is the complex structure inherited from the field dynamics by $\mathcal{M}_n$ at the adiabatic limit. The contribution of (79) is therefore the area Σ enclosed by the loop γ in $\mathcal{M}_n$. Here we do not repeat Manton's derivation of this because there are no differences in the generalized models under discussion. We observe, however, that the action of the mechanical system is of the form

$$T=\sum_{b=1}^{n}\left\{\int_{\gamma}\mathrm{d}ta_j^b[\gamma]\frac{\mathrm{d}q_b^j}{\mathrm{d}t}+\frac{2\pi\kappa}{e}\int_{\gamma}\mathrm{d}t\frac{\mathrm{d}\psi_b}{\mathrm{d}t}\right\} \quad (81)$$

where

$$a_j^b[\gamma]=\hat{a}_j^b[\gamma]+\frac{2\pi\kappa}{e}\varepsilon_{jk}b_b^k[\gamma]. \quad (82)$$

4 The generalized Abelian Higgs model

4.1 Self-dual vortices in the generalized AH model

A solvable adiabatic dynamics on the moduli space of vortices $\mathcal{M}_n$ also arises in the generalized Abelian Higgs model where the field dynamics is governed by the action:

$$S=\int\mathrm{d}^3x\left\{-\frac{1}{4}G(\varphi)F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}D_\mu\phi^*D^\mu\phi-U(\varphi)\right\}. \quad (83)$$

The system is relativistic, quadratic in time derivatives of the fields, and was proposed by Lee and Nam in [25]. Because G depends only on φ, gauge invariance is guaranteed. The model has been written in a generic form, with $G(\varphi)$ and $U(\varphi)$ unspecified; we require only that both functions be positive and definite. There are several physical situations in which this kind of model is interesting; see [25].

To identify the kinetic and potential parts of (83), we choose the temporal gauge and write the action in the form

$$S=\int\mathrm{d}t(T-V), \quad (84)$$

$$T=\frac{1}{2}\int\mathrm{d}^2x\{G(\varphi)\dot{A}_k\dot{A}_k+\dot{\phi}^*\dot{\phi}\}, \quad (85)$$

$$V=\int\mathrm{d}^2x\left\{\frac{1}{2}G(\varphi)F_{12}^2+\frac{1}{2}D_k\phi^*D_k\phi-U(\varphi)\right\}. \quad (86)$$

Observe that the Abelian Higgs model corresponds to the choice

$$G(\varphi)=1,\qquad U(\varphi)=\frac{\lambda}{8}(\varphi^2-v^2)^2. \quad (87)$$

The static energy V of the CSH model, however, is obtained by choosing

$$G(\varphi)=\frac{\kappa^2}{\mathrm{e}^2\varphi^2},\qquad U(\varphi)=\frac{\lambda}{8}\varphi^2(\varphi^2-v^2)^2, \quad (88)$$

but now the kinetic energy is different from the kinetic energy of the CSH system; as a consequence, the Gauss law derived from (83) as a constraint equation,

$$\partial_k[G(\varphi)F_{0k}]-e\mathrm{Im}(\phi D_o\phi^*)=0, \quad (89)$$

also differs from the Chern–Simons Gauss law; the electric charge is not the source of the magnetic field, and exotic statistics do not develop in this model. In any case, a configuration space $\mathcal{C}=\cup_{n\in\mathbf{Z}}\mathcal{C}_n$ corresponds to every V of the form (86), such that U gives rise to an asymmetric vacuum. Each field configuration in $\mathcal{C}_n$ has quantized magnetic flux: $e\Phi_M=2\pi n$. Furthermore, given any $G(\varphi)$ there exists a potential U that allows for self-duality equations [25]; one immediately sees that

$$U(\varphi)=\frac{\lambda\kappa^2}{8e^2G(\varphi)}(\varphi^2-v^2)^2 \quad (90)$$

produces the Bogomolny splitting

$$V=\int\mathrm{d}^2x\left\{\frac{1}{2}\left[\sqrt{G(\varphi)}F_{12}\mp\frac{e}{2\sqrt{G(\varphi)}}(\varphi^2-v^2)\right]^2 +|D_1\phi\pm\mathrm{i}D_2\phi|^2+ +\frac{1}{8}\left(\frac{\lambda\kappa^2}{\mathrm{e}^2}-\mathrm{e}^2\right)\frac{(\varphi^2-v^2)^2}{G(\varphi)}\right\}\pm\frac{ev^2}{2}\Phi_{\mathrm{M}}. \quad (91)$$

At the critical point $\lambda=e^4/\kappa^2$, the bound is saturated by the solutions of the self-duality equations

$$eF_{12}=\pm\frac{e^2v^2}{2G(\varphi)}\left(\frac{\varphi^2}{v^2}-1\right) \quad (92)$$

$$D_1\phi\pm\mathrm{i}D_2\phi=0 \quad (93)$$

that have energy $V=\pi v^2 n$ if they belong to $\mathcal{C}_n$.

As in the CSH model, the Higgs field corresponding to self-dual solutions has n zeros at the points $\vec{q}_a$ in the plane, and from (93), one sees that ϕ behaves near these zeros as in the vortex solutions of the CSH model. Away from the zeros, the equations with the upper sign take the form

$$\nabla^2 u = \frac{e^2 v^2}{G(u)}(e^u - 1), \quad (94)$$

$$eA_k = -\frac{1}{2}(\partial_k \Theta + \varepsilon_{kj}\partial_j u). \quad (95)$$

The vortex solutions of (92) and (93) are the Nielsen–Olesen (NO) vortices if we take option (87), or the Jackiw–Lee–Weinberg (JLW) vortices, if (88) is preferred. We emphasize that the same vortex equations and identical moduli spaces of solutions are shared by different physical systems; the physical nature and properties of the vortices depend crucially on the model. For instance, the NO vortices are neutral in the AH system, but electrically charged in the Ginzburg–Landau theory of [16]. By the same token, JLW vortices have electric charge in the CSH system and are neutral in the generalized AH model under discussion. Henceforth, we expect different adiabatic dynamics on the moduli space, depending on the system in question. However, we can trust the hypothesis that the moduli spaces of solutions of (92) and (93) are isomorphic for different $G(\varphi)$, because the local treatment of the moduli by means of index theorem techniques is insensitive to the form of $G(\varphi)$; see [29] or [7]. In the rest of the paper we shall admit that the zeros $\vec{q}_a$ of the Higgs field parametrize the moduli space of solutions of (92)–(93).

4.2 Second-order vortex dynamics: comparison with the AH model

In order to study the dynamics on $\mathcal{M}_n$, we start by fixing the gauge, i.e, by choosing the phase $\Theta(\vec{x};\vec{q}_a)$. The choice cannot be arbitrary; because we are working in the temporal gauge, the Gauss law

$$\partial_k(G\dot{A}_k) + \frac{1}{2}\mathrm{e}v^2\mathrm{e}^u\dot{\Theta} = 0 \quad (96)$$

must be maintained to ensure the invariance of (85) under gauge transformations with parameter $\Lambda(\vec{x};\vec{q}_a)$ varying on the moduli space. This is the main difference from the Chern–Simons theories; in this case there is no freedom to choose Θ in $\mathbf{R}^2 \times \mathcal{M}_n$. The Gauss law and the boundary conditions at the centers of the vortices and at infinity fix the gauge completely: Setting $\dot{A}_k \to \delta A_k \equiv A_k(\vec{x};\vec{q}_a + \delta\vec{q}_a) - A_k(\vec{x};\vec{q}_a)$ and using (95), we obtain from (96) the differential equation

$$\frac{\mathrm{d}G}{\mathrm{d}u}\partial_k u[\partial_k\delta\Theta + \varepsilon_{kj}\partial_j\delta u] + G\nabla^2\delta\Theta = e^2v^2e^u\delta\Theta, \quad (97)$$

which, together with the linearization of (94),

$$G(u)\nabla^2\delta u + \frac{\mathrm{d}G}{\mathrm{d}u}\nabla^2 u\delta u = e^2v^2e^u\delta u, \quad (98)$$

locally determines $\Gamma(\vec{x};\vec{q}_a)$ near each point of $\mathcal{M}_n$ and allows one to compute $\dot{u}$ and $\dot{\Theta}$ in terms of the $\dot{\vec{q}}_a$ in a definite way:

$$\dot{u}(\vec{x};\vec{q}_a) = \frac{\partial u(\vec{x};\vec{q}_a)}{\partial q_b^k}\dot{q}_b^k \quad (99)$$

$$\dot{\Theta}(\vec{x};\vec{q}_a) = \frac{\partial \Theta(\vec{x};\vec{q}_a)}{\partial q_b^k}\dot{q}_b^k \quad (100)$$

All the time derivatives in (84) can be expressed in terms of $\dot{u}$ and $\dot{\Theta}$. Because both quantities are singular at the vortex centers, it is convenient to integrate over $\tilde{\mathbf{R}}^2 = \mathbf{R}^2 - \cup_a\triangle_a$ if $\triangle_a$ is an infinitesimal disk surrounding the ath vortex. Given that even for the case $G(\varphi) \propto \varphi^{-k}, k > 1$, as happens for the CSH vortices, the integrand is regular everywhere (near an m-vortex $G \simeq r^{-mk}$ with $r = |\vec{x}-\vec{q}_a|$; but from linearization of (92), one has $\dot{A}_k \simeq r^{km-m+1}$, hence $G\dot{A}_k\dot{A}_k \simeq r^{mk-2m+1}$), eliminating these disks from the integration domain has a negligible effect on T. Now,

$$G\dot{A}_k\dot{A}_k = -\frac{1}{2e}G\dot{A}_k(\partial_k\dot{\Theta} + \varepsilon_{kj}\partial_j\dot{u}) \quad (101)$$

$$\dot{\phi}^*\dot{\phi} = \frac{v^2}{4}e^u(\dot{u}^2 + \dot{\Theta}^2), \quad (102)$$

and from the first equation, we have

$$G\dot{A}_k\dot{A}_k = -\frac{1}{2e}\partial_k[G\dot{\Theta}\dot{A}_k + \varepsilon_{jk}G\dot{u}\dot{A}_j] + \frac{\dot{\Theta}}{2e}\partial_k(G\dot{A}_k) - \frac{\dot{u}}{2e}G\dot{F}_{12} + \frac{\dot{u}\dot{A}_j}{2e}\varepsilon_{jk}\partial_k G \quad (103)$$

However, using (96) and (92), we have

$$\frac{\dot{\Theta}}{2e}\partial_k(G\dot{A}_k) = -\frac{v^2}{4}e^u\dot{\Theta}^2 \quad (104)$$

$$\frac{\dot{u}}{2e}G\dot{F}_{12} = \frac{v^2}{4}e^u\dot{u}^2 - \frac{\dot{u}\dot{G}}{2e}F_{12}, \quad (105)$$

so that the final expression for the kinetic energy is

$$T = \frac{1}{2}\int_{\tilde{\mathbf{R}}^2}\mathrm{d}^2x\left\{-\frac{1}{2e}\partial_k[G\dot{\Theta}\dot{A}_k + \varepsilon_{jk}G\dot{u}\dot{A}_j] + \frac{\dot{u}}{2e}[\dot{A}_j\varepsilon_{jk}\partial_k G - \dot{G}F_{12}]\right\}. \quad (106)$$

Note that the AH model is special: In this case, T reduces to a contour integral and can therefore be given in terms of data localized at the center of each vortex [14]. In all other cases it is necessary to integrate over all $\tilde{\mathbf{R}}^2$, which cannot be accomplished without analytical knowledge of the vorticial fields.

There is still another aspect with respect to which the AH model is special: It is the only model of the type (83) whose kinetic energy is associated with a Kähler metric on $\mathcal{M}_n$. This can be seen following the method explained in [14]. To address this point, it is convenient to replace our vectorial notation by the standard complex one: $z =$

$x^1 + \mathrm{i}x^2$, $z_a = q_a^1 + \mathrm{i}q_a^2$ and $a = A_1 - \mathrm{i}A_2$.The kinetic energy is of the form

$$T = \frac{1}{2} g_{z_a z_b} \dot{z}_a \dot{z}_b + g_{z_a z_b^*} \dot{z}_a \dot{z}_b^* + \frac{1}{2} g_{z_a^* z_b^*} \dot{z}_a^* \dot{z}_b^*. \quad (107)$$

As we already mentioned, due to (85), g cannot be expressed in closed form except in the AH model. In any case, $\mathcal{M}_n$ has a natural complex structure:

$$\begin{aligned} J : T\mathcal{M}_n &\to T\mathcal{M}_n \\ \{\dot{z}_a\} &\to \{i\dot{z}_a\}. \end{aligned} \quad (108)$$

On the other hand, from (95) and the exponential expression for ϕ, it is easy to see that

$$\dot{\phi} = \phi\eta \quad (109)$$

$$e\dot{a} = \mathrm{i}\partial_z \eta^* \quad (110)$$

where $\eta = 1/2(\dot{u} + \mathrm{i}\dot{\Theta})$, hence $T\mathcal{M}_n$ can be identified with the space of η deformations. Although the complete determination of η corresponding to some given $\dot{z}_a$ is not possible, we are at least able to write it as

$$\eta = -\sum_{a=1}^{n} \dot{z}_a \beta_a(z, z^*; z_a, z_a^*) \quad (111)$$

where

$$\beta_a(z, z^*; z_a, z_a^*) \simeq \frac{1}{z - z_a} = \frac{1}{|\vec{x} - \vec{q}_a|} \mathrm{e}^{-\mathrm{i}\theta(\vec{x} - \vec{q}_a)} \quad (112)$$

for z very close to z_a. To do this, we have used only the linearity of (97) and (98) and the regularity of $\dot{\phi}$ on all $\mathbf{R}^2$. To prove that the coefficients in (111) are precisely $\dot{z}_a$, it is enough to solve $\phi + t\dot{\phi} = 0$. It follows from (111) that the complex structure (108) is equivalent to

$$J\eta = \mathrm{i}\eta. \quad (113)$$

Now, from (85), the metric on $\mathcal{M}_n$ can be recast as

$$g(\eta_1, \eta_2) = \frac{1}{4} \int \mathrm{d}^2x \{G(\varphi)[\dot{a}_1^* \dot{a}_2 + \dot{a}_2^* \dot{a}_1] + \dot{\phi}_1^* \dot{\phi}_2 + \dot{\phi}_2^* \dot{\phi}_1\}, \quad (114)$$

where both $\dot{a}_r$ and $\dot{\phi}_r$ come from η_r by using (109, 110). Clearly g is hermitian, $g(J\eta_1, J\eta_2) = g(\eta_1, \eta_2)$, and its Kähler form $\omega(\eta_1, \eta_2) = g(J\eta_1, \eta_2)$ is

$$\omega = \frac{\mathrm{i}}{4} \int \mathrm{d}^2x \{G(\varphi)\mathrm{d}a^* \wedge \mathrm{d}a - \mathrm{d}\phi^* \wedge \mathrm{d}\phi\}. \quad (115)$$

It is easy to compute the exterior derivative of ω,

$$\mathrm{d}\omega = \frac{\mathrm{i}}{4} \int \mathrm{d}^2x \left\{ \frac{1}{2} \varphi \frac{\mathrm{d}G}{\mathrm{d}\varphi} [\mathrm{d}\eta + \mathrm{d}\eta^*] \wedge \mathrm{d}a^* \wedge \mathrm{d}a \right\} \quad (116)$$

because $\mathrm{d}\phi = \phi \mathrm{d}\eta$ is an element in $\Lambda\mathcal{M}_n$. Therefore, ω is closed only if G is constant, i.e., for the AH model.

4.3 Vortex scattering: comparison with the CSH model

From formula (106), we have seen the difficulty involved in finding an exact closed expression for T when G is not a constant. If the vortices are close enough, however, it is possible to obtain a picture of the scattering that is essentially correct. In the case of $n = 2$, the space of the polynomials $P_2(z) = (z - z_1)(z - z_2) = z^2 + a_1 z + a_2$ is isomorphic to $\mathcal{M}_2$. Notice that z_1, z_2 are the vortex centers, and $\mathcal{M}_2$ is the set of unordered pairs of points in the plane; given (a_1, a_2), we have either $(z_1 = z_+, z_2 = z_-)$ or $(z_1 = z_-, z_2 = z_+)$, where $z_\pm = (a_1 \pm \sqrt{a_1^2 - 4a_2})/2$. In the center-of-mass system, $(a_1 = 0, a_2 = w)$ implies $P_2^R(z) = (z - \sqrt{w})(z + \sqrt{w})$. The motion is symmetric around the CM, and, when $w \to 0$, the two vortices tend to overlap at the origin. Reciprocally, we can use (109) to express the scalar field of a system of two neighbouring vortices as

$$\phi(z, z^*; t) = \phi^{(2)}(z, z^*) - w(t)\phi^{(2)}(z, z^*)\beta(z, z^*). \quad (117)$$

where $w, \dot{w}$ are small, $\phi^{(2)}$ is the radial 2-vortex solution, and β, accounting for the splitting of the two vortices, behaves as $\beta \simeq 1/z^2$ near $z \simeq 0$ (see (112)). Hence, we see that $\phi(z, z^*; t) = 0$ has the symmetric roots $z_1(t) = \sqrt{w(t)}$, $z_2(t) = -\sqrt{w(t)}$ around the origin, fitting with the above description in terms of $P_2^R(z)$. From (117),

$$\dot{\phi} = -\dot{w}\phi^{(2)}\beta, \quad (118)$$

and by comparison with (109) and (110), we know that

$$e\dot{a} = \mathrm{i}\dot{w}\partial_z \beta^* \quad (119)$$

hence the kinetic energy is

$$T = \frac{1}{2}|\dot{w}|^2 \int \mathrm{d}^2x \left\{ \frac{G}{\mathrm{e}} |\partial_z \beta^*|^2 + |\phi^{(2)}\beta|^2 \right\} \equiv \frac{1}{2} M |\dot{w}|^2, \quad (120)$$

where M is given in terms of the fields of the radial 2-vortex and the deformation β coming from (97) and (98). The form of (120) as a function of the relative coordinate $z_r(t) = \sqrt{w(t)}$ is

$$T = 2M|z_r|^2 |\dot{z}_r|^2. \quad (121)$$

However, to study the movement of non-distant vortices, it is more convenient to use (120) directly. Because (120) is the kinetic energy of a free particle in the w-plane, the radial trajectories crossing the origin are solutions of the dynamics. Nevertheless, in view of the equation $z_r(t) = \sqrt{w(t)}$, we find the correspondence shown in Fig. 1, and the celebrated 90° scattering appears as a generic feature of the models (83). Note that written in terms of the w variable, which is the good coordinate in the moduli space, the metric is flat near the point where the two vortices overlap, showing that the manifold $\mathcal{M}_2$ is smooth at this point and that the conical singularity suggested by (121) is only an artifact of the wrong relative coordinate having been chosen. In fact, we do not expect that the abrupt

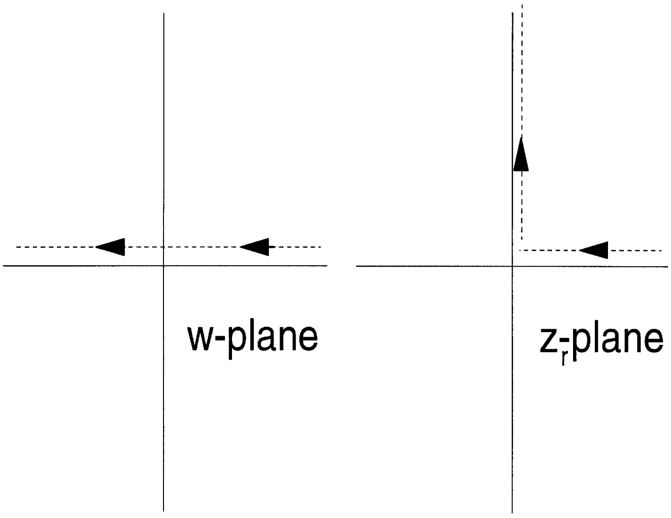

Fig. 1. Scattering of a system of two vortices, as seen from thesquared relative coordinate and from the true relative coordinate planes

change in direction shown in Fig. 1 actually occurs, the reason being that (120) is an asymptotic expression valid only for very small intervortex distances. In a realistic scattering, the initial separation between the two vortices is enough to bring the subdominant contributions not included in (120) into play. These in turn give rise to interactions that produce the smooth bending of the trajectory and the situation depicted in Fig. 1 is reached only asymptotically.

It is interesting at this point to study the vortex motion under these conditions in the CSH system, where the term

$$L^{(1)} = -\frac{2\pi\kappa}{e^2}\left[\frac{1}{q^2} - b(q)\right]\epsilon_{kj}\dot{q}^k q^j \tag{122}$$

leads to a slightly modified kinetic energy [30],

$$T = \frac{m}{2}|z_r|^2|\dot{z}_r|^2 + \frac{2\pi\kappa}{e^2}c|z_r|^4\dot{\theta}_r \tag{123}$$

where $\theta_r = \arg z_r$, $m = 2\int \mathrm{d}x^2|\phi^{(2)}\beta|^2$ and $b(q) \simeq (1/q^2) + cq^2$. The expansion of the deformation factor $b(q)$ induced by the interaction with the other vortices around the point $q = 0$ differs in the CSH system from that of the AH model. An indirect argument suggests that the tedious computation leading to such a result is correct. Unlike in the case of the Ginzburg–Landau vortices, $b(q) - (1/q^2)$ cannot be constant at $q = 0$ in the CSH model, because of the nature of the Higgs potential ruling the interactions. Again, the w-coordinate is better suited to describing the vortex motion, and we find

$$T = \frac{\mu}{2}(|\dot{w}|^2 + |w|^2\dot{\chi}^2) + \gamma|w|^2\dot{\chi}, \tag{124}$$

where $\mu = m/8$, $\gamma = \pi\kappa c/e^2$ and $w = |w|e^{i\chi}$. Although there is a term causing 90° scattering, the new linear term in $\dot{\chi}$, however, completely modifies this behaviour. An intrinsic angular momentum is induced by this term

$$J = \frac{\partial T}{\partial\dot{\chi}} = \mu|\omega|^2\dot{\chi} + \gamma|\omega|^2, \tag{125}$$

and is a constant of motion; $\dot{J} = 0$ because $\partial T/\partial\chi = 0$. The energy of this mechanical system is

$$H = \frac{1}{2}\mu|\dot{\omega}|^2 + \frac{(J^2 - \gamma|\omega|^2)^2}{2\mu|\omega|^2}, \tag{126}$$

which is equivalent to an isotropic harmonic oscillator. Choosing the constant of motion as $J = j$, we have

$$H = \frac{1}{2}\mu|\dot{\omega}|^2 + \frac{j^2}{2\mu|\omega|^2} + \frac{\gamma^2|\omega|^2}{2\mu} - \frac{j\gamma}{\mu}. \tag{127}$$

All the trajectories are thus ellipses, and the motion corresponds to bound states of two vortices orbiting around each other. This is consistent with what was discussed in Sect. 2, that the inertia of a CSH-vortex is smaller than its mass: The vortices are trapped, forming bound states as a result of the first-order dynamics. In fact, modifications due to higher-order terms in the expansion of $b(q)$, to be taken into account at larger intervortex distances, do not alter this picture. The energy and angular momentum would in this case be:

$$H = \frac{1}{2}\mu|\dot{\omega}|^2 + \frac{(J^2 - h(|\omega|)^2}{2\mu|\omega|^2}, \tag{128}$$

$$J = \mu|\omega|^2\dot{\chi} + h(|\omega|) \tag{129}$$

where $h(|\omega|)$ is a power series in $|\omega|^2$. From $\dot{H} = \dot{J} = 0$, one reads the motion equations:

$$\mu|\ddot{\omega}| - \frac{1}{\mu|\omega|^3}(J - h(|\omega|)(J - h(|\omega|) + |\omega|h'(|\omega|)) = 0 \tag{130}$$

$$\mu|\omega|\ddot{\chi} + 2\mu|\dot{\omega}|\dot{\chi} + \frac{h'(|\omega|)|\dot{\omega}|}{|\omega|} = 0. \tag{131}$$

Circular trajectories, $|\omega| = a$, occur if $j = h(a) - ah'(a)$ with angular velocity: $\dot{\chi} = -h'(a)/2a\mu$. Vortex-bound states do not arise only at short distances.

5 Conclusions and outlook

The application of the Manton approach to the low-speed dynamics of the topological vortices in the CSH model is too involved to allow a successful analytical treatment. Nevertheless, we have shown that it is possible to build two different kinds of self-dual generalized Abelian Higgs systems with solvable slow vorticial dynamics, and whose parameters can eventually be adjusted to obtain exactly the CSH moduli space. Remarkably enough, despite important differences in their field profiles, the qualitative dynamical behavior of the vortices in each class of generalized systems is not particularly model-dependent, but generic: All the possible non-relativistic first-order systems give rise to a uniform circular motion of the vortices around the barycenter, and for all the relativistic second-order ones, the head-on collision of two defects leads to right-angle scattering. It is believed that the dynamics of the original CSH vortices results from some entanglement of these two effects. A few final words on quantization. For the quadratic model of Sect. 4, the transition from classical to quantum mechanics is straightforward: The Laplace–Beltrami operator corresponding to the metric on the moduli space becomes the quantum Hamiltonian, replacing the classical kinetic energy as generator of the dynamics. In the linear model of Sect. 3 things are more interesting (less standard), especially when the charged background is incorporated. Observe that (81) is no more than topological classical mechanics associated with the space of paths in $\mathcal{M}_n$; see [31]. The quantization is almost trivial when $\mathcal{M}_n$ is topologically trivial. The Hilbert space reduces to the ground state, which is degenerated; e.g., if $b_b^k[\gamma] = 0$, it would be the first Landau level. If vortices move in a compact space, a two-sphere for instance, things become more difficult, and one would need to consider the Floer homology of the symplectic compact manifold $\mathcal{M}_n$ [32].

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